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UNDERSTANDING THE COMPLEXITY IN LOW DIMENSIONAL SYSTEMS

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Abstract. *Complex System is any system that presents involved behavior, and is hard to be modeled by using the reductionist approach of successive subdivision, searching for "elementary" constituents. Nature provides us with plenty of examples of these systems, in fields as diverse as biology, chemistry, geology, physics, and fluid mechanics, and engineering. What happens, in general, is that for these systems we have a situation where a large number of both attracting and unstable chaotic sets coexist. As a result, we can have a rich and varied dynamical behavior, where many competing behaviors coexist. In this work, we present and discuss simple mechanical systems that are nice paradigms of Complex System, when they are subjected to random external noise. We argue that systems with few degrees of freedom can present the same complex behavior under quite general conditions.*

Palavras-chave: *Complexity, Chaos, Multistability.*

1. INTRODUCTION

Complexity has become an issue of recent great interest (badii97). It is associated to systems that presents involved behavior, and is hard to model using the reducionist approach of successive subdivision, searching for "elementary constituents" (macau99). While there is so far no general definition of complexity, previous works suggest that we might expect to find the following characteristics concerning the system's behavior of a complex system (grassberger91,crutchfield89,horgan95,poon95): (i) a behavior that is neither completely ordered and predictable nor completely random and umpredictable; (ii) its evolution reveals patterns in which coherent structures develop at various scales, but do not exhibit elementary interconnections; (iii) the structures can show a hierarchical relationship, i.e., nontrivial structures over a wide range of scales can appear. Complex systems are very common in many natural systems such as the Rayleigh-Bérnard convection (berge83), Belousov-Zhabotinsky reaction (muller89), neuronal activity (rapp94), extended nonlinear optical systems (arecchi90), fluidized beds (daw95) etc.

Usually, the features that are typical of a Complex System appear in systems with many degrees of freedom (poon95). This is the case of all the systems previously cited. What happens, in general, is that for these systems we have a situation where a large number of both attracting and unstable chaotic sets coexist. As a result, we can have a rich and varied dynamical behavior, where many competing behaviors can exist. When the system is evolving in the neighborhood of an attracting periodic set, it will exhibit an "ordered" behavior. This behavior changes to an apparently ‘‘non-ordered” behavior when the system is evolving about the unstable sets. Thereby, the attractors themselves are responsible for the appearance of coherent structures, while the specific characteristics of each individual attractor, combined with its location relatively to the unstable sets are responsible for the appearance of a hierarchy of structures.

Recent works showed that complexity can also appear in low dimensional systems (macau99,poon95,feudel97). Both the double rotor (poon95,feudel98), which is four dimensional, and the single rotor (macau99,feudel97), which is two dimensional, under well established conditions, can be view as nice paradigms of a complex system, when they are subjected to a random noise. The key to understand how the complexity thrives in low dimensional system is the multistability phenomena. Multistability is characterized by a large number of coexisting attractors, mainly periodic ones, for a fixed set of parameters. A multistable system presents (feudel97) complicated basin structures with invariant chaotic set embedded in the fractal basin boundaries. With the exception of small open neighborhoods about the periodic attractors, the phase space is permeated by the fractal basin boundaries the dimension of which are very close to the dimension of the phase space. Though the trajectory can spend arbitrarily long times in the neighborhood of one of the stable periodic behaviors, the external noise applied to the system prevents the trajectories from settling permanently into any one of them. Thus, this system presents the same typical behavior of a complex system.

The purpose of this article is to understand multistability and its interconnections with complex systems. To accomplish that, we show how multistability thrives in conservative low dimensional chaotic system and how the complexity appears in multistable system in the presence of random noise. Furthermore we characterize low dimensional complex systems by evidencing its fundamental dynamical properties. Therefore, in Sec. II we describe the basic features of multistable systems, which are obtained from conservative ones by adding a small mount of damping. In Sec. III we focus on low dimensional complex systems and in its fundamental properties. In the last section we present general comments.

2. DISSIPATIVE SYSTEMS AND MULTISTABILITY

A conservative system, where the modulus of the determinant of the Jacobian matrix for a map is equal to 1, presents two different types of dynamics, regular and chaotic (lichtemberg92). The regular behavior occurs in the Kolmogorov-Arnold-Moser (KAM) islands and in the KAM tori, which are embedded in the chaotic sea. These islands are associated with marginally stable periodic orbits whose eigenvalues are equal to one in absolute value. The large ones, the so-called primary islands, are surrounded by smaller secondary islands. This scenario changes if a small amount of damping is added to such systems (feudel97): a family of dissipative dynamical systems appears, in which the previously marginally stable periodic orbits turn into periodic attractors whose eigenvalues are smaller than one in absolute value. Furthermore, instead of an infinite number of attractor, only a finite number of them can be found. Every previous KAM island is converted into an attractor, but the number of attractors depends on the damping level and the particular family under consideration.

There are families of typical dynamical systems where the conservative element of each family has one or, at most just a few, primary island surrounded by secondary islands. This family shows multistable behavior when small damping is added, but the number of coexisting attractors is not so large. One example of this type of family is the Hènon map, written in the following form:

$$x_{n+1} = A - x_n^2 - (1-\nu)\, y_n \qquad (1)$$
$$y_{n+1} = x_n \,.$$

This family posses two parameters. The parameter ν, which varies between 0 and 1, is the damping. In the limit ν=1, the two equations in (1) are decoupled, yielding the quadratic map. In the other limit, ν=0, the map is no longer dissipative, the determinant of the Jacobian matrix is equal to 1. The parameter *A* represents the bifurcation or control parameter and it is the nonlinearity parameter. While in the case of the quadratic map there exists only one bounded attractor over a wide range of the parameter *A*, there are several coexisting attractor for values of ν close to the no damping limit ν=0, as shown in Fig. (1).

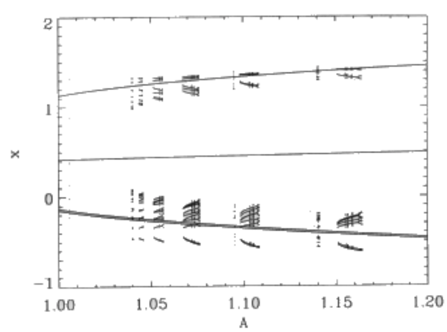

Figure 1: Biffurcation diagram for the Hénon Map with ν=0.02

There is other family of dynamical systems, in which the number of attractors associated with the primary islands scales as 1/damping, *i.e.,* when the damping tends to zero their number tends to infinity. An example of such a family of systems is the kicked singled rotor, which describes the time evolution of a mechanical pendulum that is being kicked at times *nT, n=1,2,…* , with a constant force f_0. From the differential equation of this mechanical system one can derive a map which is related to the state of the system just after each successive kick (schmidt85):

$$x_{n+1} = x_n + y_n \pmod{2\pi}$$
$$y_{n+1} = (1-\nu)\times y_n + f_0 \sin(x_n + y_n), \quad (2)$$

where x corresponds to the phase and y to the angular velocity. f_0 is the force parameter, and ν is the damping parameter, measuring the energy dissipation of the system. The parameter ν varies between 0, for a Hamiltonian situation, with no damping, and 1, in the case of a very strong damping. The dynamics lies on the cylinder $[0,2\pi)\times\Re$. In the very strong damping ν=1 limit, the system reduces to a one-dimensional circle map with a zero rotation number, and it exhibits the Feigenbaum scenario to chaos (schmidt85). The dynamics lies on the circle $[0,2\pi)$.

In the no damping case (ν=0), we have the area-preserving standard map, which was studied by Chirikov (chirikov79) and by many other authors (greene79,schmidt80,lichtemberg92,meiss83). It has stable and unstable periodic orbits, Kolmogorov-Arnold-Moser (KAM) surfaces, and chaotic regions. Depending on the nonlinear parameter f_0, the regions of regular motion and the regions of chaotic motion are complexly interwoven. As the second equation of the map is also taken to be modulo 2π, the map of the cylinder reduces now to the map of the torus $[0, 2\pi)\times[0,2\pi)$ to itself. As a consequence, each of the periodic orbits represents, in fact, a family of overlapping periodic orbits in which the velocity y differs by integer multiples of 2π. Due to the modulo 2π, all periodic orbits of a same family are located at the same location on the thorus.

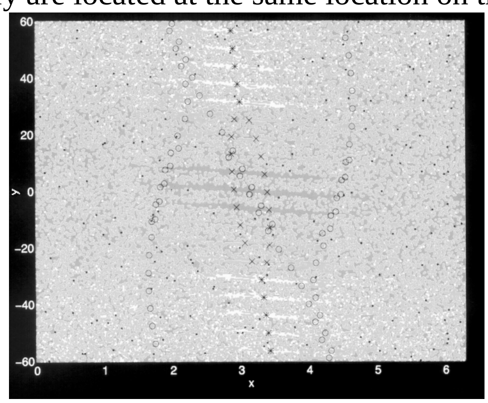

Figure 2: Basin of attraction for the single rotor with noise. * indicates the position of attracting period-one points; *x* the position of attracting period-two orbits, and *o* the position of attracting period-three orbits. (f_0 = 4.0 e ν=0.02).

If we now consider the no damping case but introduce a very small amount of dissipation (ν close to zero), the symmetry in y is broken, and the motion again takes place on the cylinder $[0,2\pi)\times\Re$. The periodic orbits become sinks and the chaotic Hamiltonian sets become unstable chaotic sets embedded in the basin boundaries separating the various sinks. The chaotic motion is hence replaced by long chaotic transients that occur before the trajectory is eventually asymptotic to one of the sinks (feudel96). Furthermore, the dissipation leads to a separation of the overlapping periodic orbits, which belong to a given family, with increasing modulo of the velocities on the cylinder. However, there is a bounded cylinder which contains all of the attractors (feudel96). This cylinder is given as

$[0, 2\pi)\times[-y_{max}, y_{max}]$, where $y_{max}=f_0/\nu$, and all trajectories are eventually trapped inside this region (feudel96). Consequently, for values of ν close to zero, there is a large, but finite, number of coexisting periodic orbits of increasing period, as can be seen in Fig. (2).

Besides a large number of coexisting attractors and unstable chaotic sets embedded in the basin boundaries, there are other general properties which are common to these families of multistable dynamical systems. The multistability property is dominated by periodic attractors, which means that the long-term behavior is regular. Just very few initial conditions, eventually located in a set of measure almost zero, asymptote to a chaotic behavior. Thus, chaotic attractors occur only rarely. However, even if apparently there are no chaotic attractors present, this does not mean that chaos is absent. In fact, there are extended chaotic sets, but unstable, lying in the basin boundaries separating the various periodic attractors. Fig. (3) shows a typical basin of attraction for a periodic attracting orbit of the single rotor. The black points are attracted to the specific attractor. The particular picture shows the basin of attraction for a fixed point at $y=6\pi$. The basins of attraction have fractal boundaries. Feudel et al. (feudel96) calculated the *uncertainty exponent* (α) which measures the sensitivity of the final state to small changes in the initial conditions. This exponent is typically related to the box counting dimension d of the basin boundary by $\alpha=D-d$, where D is the dimension of the state space. For damping $\nu=0.05$, the result is $\alpha=0.00641$, which implies $d=1.99359$; for $\nu=0.02$, the result is $\alpha=0.001$, and $d=1.999$. This means that the dimension of the basin boundaries is nearly the dimension of the state space, and they are organized in a complexly interwoven structure, with chaotic saddles embedded in these basin boundaries (grebogi88). Furthermore, extremely small changes in the initial conditions may shift a trajectory from one basin to another, which means that the system has high sensitivity to the final state. Thus, which attractor is eventually reached by a trajectory of the system depends strongly on the initial conditions which is the typical behavior of multistable systems. In this scenario, typical trajectories, starting with arbitrary initial conditions, experience periods of long chaotic transients before approaching one of the periodic attractors.

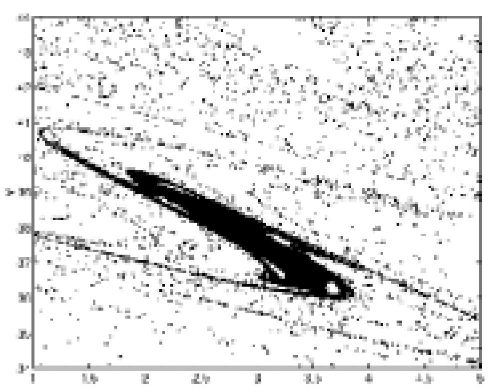

Figure 3: Enlargement of the basin of attraction for a period one attracting orbit.

Other general characteristic of multistable systems is the fact that most of the initial conditions converge to periodic attractors with low periods. This happens because the attractors with high periods have very small basins of attraction (feudel97), as can be seen on Fig. (2) for the single rotor. Thus, high periodic orbits are difficult to detect.

3. FROM MULTISTABILITY TO COMPLEXITY

Let we now understand how complexity arrives in a original multistable system. We consider the single rotor family in the presence of a small amplitude noise. The noise may prevent the trajectories from settling into any of the stable periodic behaviors (macau99,poon95). The trajectory may come close to one of the periodic attractors, and remain in its neighborhood for some time. During this period, the trajectory's behavior is governed by the periodic attractor and it is, as a consequence, ordered. However, this ordered behavior just persists for a while, because noise will eventually move the trajectory out of this "metastable" state into the fractal boundary region. In the neighborhood of fractal basin boundaries, the trajectory's behavior is governed by the unstable invariant chaotic sets that are embedded there.

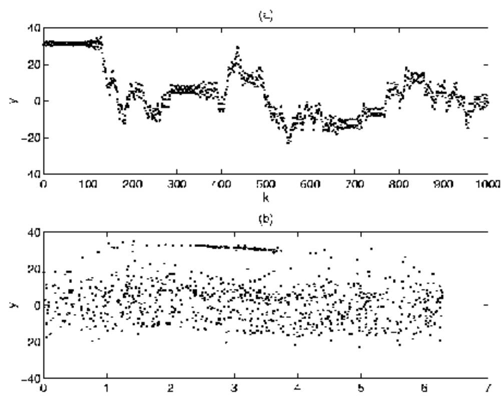

Figure 4: (a) The *y* variable, which represents the angular velocity of the noise kicked single rotor, versus the iterationnumber *n* for a typical noisy trajectory. (b) The same noisy trajectory plotted in phase space.

As a consequence, the trajectory experiments a chaotic transient behavior for some time, until it approaches the same or another periodic attractor. The period of time that the trajectory stays at the fractal boundaries corresponds to the trajectory's ``random'' behavior. Therefore, in a single rotor with noise, a typical trajectory alternates between intervals of random or chaotic motion and intervals of nearly periodic behavior. Figure 4 shows this behavior for a typical trajectory. Such behavior, which stresses the fact that the system is neither completely ordered and predictable nor completely random and unpredictable, has also been observed experimentally in Rayleigh-Benard convection (berge83), in coupled laser systems (arecchi90), and in fluidized beds (daw95). In Fig. (4), we also see that the trajectory visits the neighborhoods of different attractors in a ``random'' way. It is not possible to devise, for example, an empirical rule which allows one to forecast the sequence of attractors that will be visited by the noisy trajectory from the knowledge of the attractors previously visited. This is another typical characteristic of this system.

The evolution of an ensemble of initial conditions in physical space reveals coherent structures, as can be seen in Fig. (5). This figure is obtained by following the evolution of an ensemble of initial conditions in physical space for the single rotor with noise. We iterate this ensemble of initial conditions n times and then verify how close each of the *nth* iterated initial condition of the ensemble is from a periodic attractor. A natural number was attributed to identify periodic attractors of the system. We consider that the *nth* iterated distance between this *nth* iterated point and the periodic orbit is less than a pre-specify limiting distance d_{lim}. If we determine that the *nth* iterated point is in the neighborhood of a periodic orbit, we associate to this point a positive real number. The integer part of this number corresponds to the natural number that is attributed to the periodic orbit. The fractional part is the distance from the point to the periodic orbit normalized by d_{lim}. That positive real number is assigned to the initial condition of the ensemble corresponding to this *nth* iterated point. The picture that is showed in Fig. (5) is gotten by associating of a "color-map" to the numbers that are attributed to each initial condition of the ensemble. With this association we can unveil which initial conditions fall in a given coherent structure after n iterates and which ones are ejected from the coherent structure into the random structure. Thus, in Fig. (5), regions with the same hue indicate which initial points will be after n iterations in the neighborhood of the same periodic attractor, while the saturation of each point in the region indicates how close its *nth* iteration will be from the periodic attractor. We should mention that the average number of iterations a trajectory spends in a coherent structure decreases with the noise level.

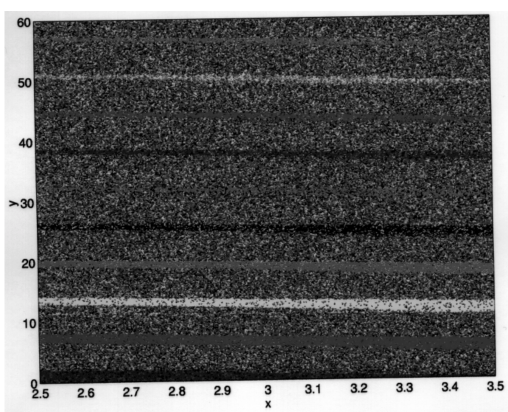

Figure 5: Coherent structures that appears as a result of the evolution of an ensemble of initial conditions in physical space.

The appearance of coherent structures are associated with the ordered behavior of the trajectories in the neighborhood of the periodic attractors. We can quantify the amount of order and randomness associated with the sequence in which successive periodic attractors are visited, as the system dynamically evolves, by using the *Kolmogorov-Sinai (KS) entropy* (kolmogorov58,sinai59). Its calculation requires the attribution of a partition to the phase space, and it quantifies the average uncertainty per time step about the partition element that is currently visited by the orbit. We choose a natural disjoint partition of the phase space by associating each disjoint partition to the neighborhood of one or more of the periodic attractors. An alphabet with ten symbols is then introduced to symbolic identify the partitions. Consequently, the evolution of a trajectory of the system can be represented by a sequence of the symbols of the alphabet in accordance with the sequence of the partitions that the trajectory visits. We compute then the Kolmogorov-Sinai (KS) entropy using the relation (adler65,cornfeld82):

$$h = \lim_{n\to\infty} \frac{H_n}{n} = \lim_{n\to\infty} \frac{1}{n}\left(-\sum_{|S|=n} p(S)\ln\big(p(S)\big) \right), \quad (3)$$

where $S=s_1s_2...s_n$ denotes a finite symbol sequence that is associated with the occurrence of a trajectory that successively and sequentially visits the partitions corresponding to the symbols $s_1, s_2, ..., s_n$, while $p(S)$ is the joint probability for the occurrence of this sequence S. Our calculation shows that H_n/n converges to the value h ~1.65. If we recall that the KS entropy value would be ln (10) ~2.30 if the sequence were completely uncorrelated, and zero if the sequence were periodic, we conclude that the sequence is neither predictable nor completely unpredictable. Reference (poon95) suggested that an intermediate value of KS indicates the existence of structure in the set of all possible sequences. We associated this intermediate entropy result as an indication that the coherent structures associated with the evolution of the system in the neighborhood of periodic attractors of different periodicity (scale) do not exhibit elementary or simple interconnections.

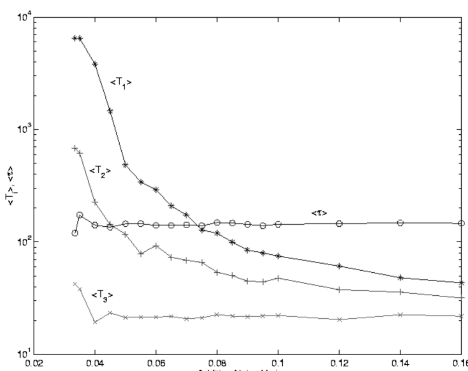

Figure 6: The mean escape times T_i for some of the attractors and the average length of the chaotic transient τ for different vaules of noise amplitude

This complexly interwoven interconnection between the coherent or periodic and random or chaotic structures reflects the appearance of nontrivial time scales in the system. In Fig. (6) we have the mean escape times $<T_i>$ for some of the attractors and the average length of the chaotic transient $<\tau>$ associated with the random structure for different values of noise amplitude. The mean escape time is, in general, different for different attractors, for the same noise amplitude, but it does exist a unique scale law applicable to all the attractors for the relation between the escape time and the noise amplitude. The escape time is exponentially distributed, though the decay rate is different for different attractors. The average length of the chaotic transient $<\tau>$ is related to the dimension and the Lyapunov exponents of the chaotic saddles that are embedded in the fractal basin boundary (kantz85,hsu88,grebogi86). It is a result of the contribution of all chaotic saddles embedded in the boundary, which, in general, individually each has a distinct time scale.

As a result of our discussion, we conclude that the single rotor with noise is a system which presents the following characteristics: (i) its behavior is neither completely ordered and predictable nor completely random and unpredictable; (ii) its time evolution reveals patterns and structures over various time and spatial scales; (iii) this pattern forms hierarchies, *i.e.*, nontrivial structures over a wide range of scales, and the interconnection among the structures is complicated. It means that the single rotor with noise can be characterized as a complex system (badii97), regardless for the fact that is a system of low (just two!) dimension. The same conclusion follows when similar arguments are applied to other families of multistable systems.

4. CONCLUSION

We have shown how conservative systems with a small amount of dissipation can display the rich dynamical behavior that is characteristic of multistable system. In that system, the dynamics is dominated by a large number of coexisting periodic attractors; high-periodic attractors have very small basins of attractions; the basins of attractions of the coexisting attractors are complexly interwoven; the estimated box dimension of the basin boundaries are close to that of the state space. Furthermore, we have shown how complexity can be characterized in low dimensional systems and how it thrives if a random noise is present in multistable system.

REFERÊNCIAS

Adler. R. C., Konheim, A. C., and McAndrew, M. H., 1965, Topological Entropy, Trans. Am. Math Soc., vol. 114, p. 309.

Arecchi, F. T., Giacomelli, G., Ramazza, P. L., and Residori, S., 1990, Experimental evidence of chaotic itinerancy and spatiotemporal chaos in optics, Phys. Rev. Lett., vol. 65, pp. 2531-2534.

Badii, R. & Politi, A., 1997, Complexity: Hierarchical Structures and Scaling in Physics Cambridge.

Bergé, P. & Dubois, M., 1983,Transient reemergent order in convective spatial chaos, Phys. Lett. A, vol. 93, p. 365.

Chirikov, B. V., 1979, A universal instability of many-dimensional oscillator systems, Phys. Rep., vol. 52, p. 263.

Cornfeld, I., Fomin, S., and Sinai, Ya. G., 1982, Ergodic Theory, Springer-Verlag, New York.

Crutchfield, J. P. & Young, K., 1989, Inferring Statistical Complexity, Phys. Rev. Lett. Vol.63, pp. 105-108.

Daw, C. S., Finney, C. E. A., Vasudevan, M., van Goor, N. A., Nguyen, K., Bruns, D. D., Kostelich, E. J., Grebogi, C., Ott, E., and Yorke, J. A., 1995, Self-organization and chaos in a fluidized bed, Phys Rev Lett., vol. 75, pp. 2308-2311.

Feudel, U. & Grebogi, C., 1997, Multistability and the control of complexity, Chaos, vol. 7, pp. 597-604.
Feudel, U., Grebogi, C., Hunt, B., and Yorke, J. A., 1996, Map with more than 100 coexisting low-period periodic attractors, Phys. Rev. E, vol. 54, pp. 71-81.
Feudel, U., Grebogi, C., Poon, L., and Yorke, J. A., 1998, Dynamical properties of a simple mechanical system with a large number of coexisting periodic attractors, Chaos, Solitrons & Fractals, vol. 9, pp. 171-180.
Grassberger, P., 1991, Randomness, information and complexity, in Proceedings of Fifth Mexican School on Statistical Mechanics, edited by F. Ramos-Gomez, World Scientific, Singapore, p. 59.
Grebogi C., Ott E., and Yorke, J. A., 1986, Metamorphoses of basin boundaries in nonlinear dynamical systems, Phys Rev. Lett, vol. 56, pp. 1011-1014.
Grebogi, C., Nusse, H., Ott, E., and Yorke, J. A., 1988, Basis sets: sets that determine the dimensions of basin boundaries, in Lecture Notes in Mathematics (Dynamical Systems), vol. 1342, edited by J. C. Alexander, Springer-Verlag, New York, pp. 220-250.
Greene, J. M., 1979, A method for determining a stochastic transition, J. Math Phys., vol. 20, p. 1183.
Horgan, J.,1995, From complexity to perplexity, Sci. American. Vol. 272, No. 6, pp. 104.
Hsu, G. -H, Ott, E., and Grebogi, C., 1988, Strange saddles and the dimension of their invariant manifolds, Phys. Lett. A, vol. 127, pp. 199-204.
Kantz, H. & Grassberger, P., 1985, Repellers, semi-attractors, and long-lived chaotic transients, Physica D (Amsterdam, vol.17, pp. 75-86.
Kolmogorov, A. N., 1958, Eine neue metrische Invariante transitiver dynamischer Systeme und Automorphismen von Lebesque-Raumen (in russ.), Dok. Akad. Nauk SSSR, vol. 119, p. 861.
Lichtenberg, A. J. & Lieberman, M. A., 1992, Regular and chaotic dynamics, Springer-Verlag, New York.
Macau, E. & Grebogi, C., 1999, Driving trajectories in complex systems, Phys. Rev. E, vol. 59, pp. 4062-4069.
Meiss, J. D., Lary, J. R., Crawford, J. D., Grebogi, C., Kaufway, A. N., and H. D. I. Abrabanel, H. D. I, 1983, Correlations of periodic area-preserving maps, Physica D (Amsterdam), vol. 6, p. 375.
Muller, S. C. & Hess, B., 1989, in Cell to cell signalling: from experiments to theoretical models, edited by A. Goldbeter, Academic Press, London, p. 503.
Poon, L. & Grebogi, C., 1995, Controlling Complexity, Phys. Rev. Lett., vol. 75, pp. 4023-4026.
Rapp, P. E., Zimmerman, I. D., Vining, E. P., and Cohen, N., 1994, The algorithmic complexity of neural spike trains increases during focal seizures, J. Neuroscience, vol. 14, p. 4731.
Schmidt, G. & B. W. Wang, B. W., 1985, Dissipative standard map, Phys. Rev. A, vol. 32, pp. 2994-2999.
Schmidt, G., 1980, Stochasticity and fixed-point transitions, Phys. Rev. A vol. 22, pp. 2849-2854.
Ya. G. Sinai, Ya. G., 1959, The notion of entropy of a dynamical system, Dok. Akad. Nauk SSSR vol. 124, p.768.